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Coalescence of Matrices, Regularity and Singularity of Birkhoff Interpolation Problems

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1. INTRODUCTION

The Birkhoff interpolation problem [1] discussed here is the problem of finding a polynomial $P_n(x) = \sum_{k=0}^{n} a_k x^k/k!$ which satisfies conditions

$$P_n^{(k)}(x_i) = c_{ik} \qquad (e_{ik} = 1). \tag{1.1}$$

The *incidence matrix* $E = (e_{ik})_{i=1}^{m}{}_{k=0}^{n}$ is an $m \times (n+1)$ matrix with precisely $(n+1)$ ones; $X = (x_1, ..., x_m)$ is a set of distinct knots; c_{ik} are the data. The matrix E is *regular*, if problem (1.1) has a unique solution for each selection of the set X (and of data c_{ik}). This happens if and only if the determinant $D_E(X)$ of the system (1.1) never vanishes. One distinguishes *order-regularity* (or simply *regularity*), when the x_i are real and subject to the conditions $x_1 < \cdots < x_m$, and *real* (or *complex*) *regularity*, when the x_i are arbitrary distinct real (or complex) numbers. If the determinant $D_E(X)$ vanishes for some X, the matrix E is *singular*; it is *strongly singular* when X is real and the determinant changes sign.

The Pólya functions of E are defined by

$$m(k) = \sum_{i=1}^{m} e_{ik}; \qquad M(k) = \sum_{l=0}^{k} m(l), \qquad k = 0, ..., n. \tag{1.2}$$

If

$$M(k) \geq k + 1, \qquad k = 0, ..., n-1, \tag{1.3}$$

E is called a *Pólya matrix*; it is a *Birkhoff matrix* if the stronger condition is satisfied:

$$M_k \geq k + 2, \qquad k = 0, ..., n-1. \tag{1.4}$$

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If E is a Pólya matrix, there exists a finest vertical decomposition of E into Pólya matrices:

$$E = E_1 \oplus \cdots \oplus E_u ; \tag{1.5}$$

it is called the *canonical decomposition* of E. It consists of Birkhoff matrices, and matrices with one column.

One of the important tools in the investigation of regularity is the coalescence of rows of E, introduced by Karlin and Karon [3] and studied further by Lorentz and Zeller [7] and Lorentz [4] (see also [5, 6]). In this paper we want to put this method on a broad basis, which allows multiple coalescence (Section 3). A central role is played here by the associativity of multiple coalescence. This holds for the coalescence of matrices (formula (3.1.2)), but not for their determinants. Using this fact, we obtain new criteria of singularity (Theorems 2, 3). In particular, we find a new phenomenon: Three rows of a matrix "can be so bad" that the matrix is singular for all possible variations of the remaining rows.

2. Leveling Functions

2.1. In what follows, $m(k)$ will always denote a nonnegative function with integral values, defined for $k = 0,..., n$. We use m_k interchangeably with $m(k)$, and denote by $M(k) = M_k$ the sum of m, $M(k) = \sum_{l=0}^{k} m(l)$. Let g, G always denote functions of this type with $0 \leqslant g(k) \leqslant 1$, $k = 0,..., n$.

For a given m, we define the function $m^0(k) = m_k^0$ with values 0 or 1 by induction. We put $m_0^0 = 1$ if and only if $m_0 \geqslant 1$. If m_l^0 and hence M_l^0 are known for $l = 0,..., k$, we define $m_{k+1}^0 = 1$ if and only if $M_{k+1} - M_k^0 \geqslant 1$. It is clear that m^0 satisfies $m_k^0 \leqslant m_k$, hence $M_k^0 \leqslant M_k$, $k = 0,..., n$.

Another way to define the function m^0 is as follows. Let

$$\mu_k = (\cdots((m_0 - 1)_+ + m_1 - 1)_+ \cdots + m_{k-1} - 1)_+ + m_k, \qquad k = 0,..., n. \tag{2.1.1}$$

Then $m_k^0 = 1$ if and only if $\mu_k \geqslant 1$. Indeed, from the definition of m^0,

$$\begin{aligned}(M_k - M_{k-1}^0 - 1)_+ + m_{k+1} &= M_k - M_{k-1}^0 - m_k^0 + m_{k+1} \\ &= M_{k+1} - M_k^0, \qquad k = 0,..., n.\end{aligned} \tag{2.1.2}$$

From this we obtain the relation

$$\mu_{k+1} = M_{k+1} - M_k^0. \tag{2.1.3}$$

For $k = 0$ this is immediate, and in general follows by induction, using (2.1.2).

Here is a corollary of this:

Let $0 \le n_1 < n$, and $M(n_1) = M^0(n_1)$. Then $\mu_{n_1} = m_{n_1}$, and for $k < n_1$,

$$\mu_k = (\cdots(m_{n_1} - 1)_+ + \cdots + m_{k+1} - 1)_+ + m_k .$$

This means that in our case, $\mu_k = m_k^0$ for $k \le n_1$ can be computed by means of the values of m_k in the interval $n_1 \le k \le n$.

Lemma 1. *For a function G with $0 \le g(l) \le 1$, let $G(l) \le M_1(l) + M_2(l)$. then also $G(l) \le M_1^0(l) + M_2(l)$, $l = 0, \ldots, n$.*

Proof. The inequality in question is certainly true for $l = 0$. Assume that it holds for $l = 0, \ldots, k$. If $m_1^0(k+1) = 1$, then $G(k+1) \le G(k) + 1 \le M_1^0(k) + M_2(k) + 1 \le M_1^0(k+1) + M_2(k+1)$. If, on the other hand $m_1^0(k+1) = 0$, then $M_1(k+1) = M_1^0(k)$ and

$$G(k+1) \le M_1(k+1) + M_2(k+1) = M_1^0(k) + M_2(k+1).$$

The *level function* of a function m is defined to be a function g, satisfying

$$0 \le g(k) \le 1, \qquad G(k) \le M(k), \qquad k = 0, \ldots, n. \tag{2.1.4}$$

for which $G(k)$ is largest possible for each k.

Proposition 1. *The level function of m exists, is unique, and coincides with m^0 defined above.*

This follows from the lemma by taking $M_1 = M$, $M_2 = 0$.

We obtain now:

Proposition 2.

$$(M_1 + M_2)^0 = (M_1^0 + M_2)^0 = (M_1^0 + M_2^0)^0. \tag{2.1.5}$$

As a corollary we have the "associativity formula"

$$((M_1 + M_2)^0 + M_3)^0 = (M_1 + (M_2 + M_3)^0)^0 = (M_1 + M_2 + M_3)^0. \tag{2.1.6}$$

2.2. For a function $m(k) = m_k$, $k = 0, \ldots, n$, with nonnegative integral values, we define the *coefficient of collision*

$$\alpha(M) = \sum_{k=0}^{n} \rho_k , \qquad \text{where} \qquad \rho_k = M_k - M_k^0. \tag{2.2.1}$$

The relation $\alpha(M) = 0$ means that $m = m^0$, or that $0 \leqslant m_k \leqslant 1$. The function M is in collision, $\alpha(M) \geqslant 1$, if and only if for some k, $M_k > M_k{}^0$, or, equivalently, if and only if for some k, $m_k \geqslant 2$.

PROPOSITION 3. *One has*

$$\alpha(M_1 + M_2) = \alpha(M_1) + \alpha(M_1{}^0 + M_2)$$
$$= \alpha(M_1) + \alpha(M_2) + \alpha(M_1{}^0 + M_2{}^0). \tag{2.2.2}$$

Proof. By (2.2.1) and Proposition 2,

$$\alpha(M_1 + M_2) = \sum_{k=0}^{n} \{M_1(k) + M_2(k) - (M_1 + M_2)^0(k)\}$$

$$= \sum_{k=0}^{n} \{M_1(k) - M_1{}^0(k)\}$$

$$+ \sum_{k=0}^{n} \{M_1{}^0(k) + M_2(k) - (M_1{}^0 + M_2)^0(k)\}.$$

2.3. An important property which a function M may have is the inequality

$$\sum_{l=k}^{n} m_l \leqslant n - k + 1, \qquad k = 0, 1, \ldots, n. \tag{*}$$

(This implies, in particular, $m_n \leqslant 1$.)

If $M_n = n + 1$, then (*) is equivalent to the "Pólya property" $M_k \geqslant k + 1$, $k = 0, \ldots, n - 1$. In this case, in (2.1.1) we have $\mu_k = M_k - k$, and consequently $m_k{}^0 = 1$, $k = 0, \ldots, n$.

If for some $0 \leqslant n_1 \leqslant n$, $\sum_{n_1}^{n} m_l = n - n_1 + 1$, then by subtraction we obtain that the restriction of m to the interval $[0, n_1 - 1]$ satisfies (*).

Let M be a function with the property (*), and let $m_k \geqslant 1$ for some $0 \leqslant k < n$. The *shift* $k \to k + 1$ transforms M into the function $\overline{M} = \Lambda M$, given by $\overline{M}_k = M_k - 1$, $\overline{M}_l = M_l$, $l \neq k$. (In Section 3, we shall adopt a slightly different definition of a shift of a row in a matrix.)

Obviously, if $G_l \leqslant \overline{M}_l$, then also $G_l \leqslant M_l$. Therefore $\overline{M}^0 \leqslant M^0$, and from (2.2.1) we obtain

$$\alpha(\overline{M}) \geqslant \alpha(M) - 1. \tag{2.3.1}$$

It can happen that $\alpha(\overline{M}) > \alpha(M)$, but most important for us will be the *reducing shifts*, for which $\alpha(\overline{M}) = \alpha(M) - 1$, or, equivalently, $\overline{M}^0 = M^0$.

A function M, with $\alpha(M) > 0$, which satisfies (*) has reducing shifts. There are k for which $m_k \geq 2$, and clearly $k \to k+1$ is a reducing shift.

A multiple shift $\tilde{\Lambda}$ of order β is a product of β shifts: $\tilde{\Lambda} = \Lambda_1 \cdots \Lambda_\beta$. If $\beta > \alpha(M)$, then $\tilde{\Lambda}$ is in collision: $\alpha(\tilde{\Lambda} M) < 0$. On the other hand, if $\beta = \alpha(M)$, then $\tilde{\Lambda} M$ is without collision if and only if each Λ_j is reducing. One can take the Λ_j to be of the special kind described above.

PROPOSITION 4. *If the function M satisfies (*), then $M_n = M_n^0$; in addition, $M_{n_1} = M^0_{n_1}$ for each n_1 with the property that M satisfies (*) on $[0, n_1]$.*

Proof. To M in $[0, n_1]$ we apply one of the "special shifts" Λ ($k \to k+1$, with $m_k \geq 2$). Then $\Lambda M = \bar{M}$ will also satisfy (*) in $[0, n_1]$. We have only to show that $\sum_{k+1}^{n_1} \bar{m}_l \leq n_1 - k$. But since $m_k \geq 2$ and $\sum_k^{n_1} m_l \leq n_1 - k + 1$, we have the required inequality.

Obviously $\bar{M}_{n_1} = M_{n_1}$. To $\bar{M}$ we again apply a special shift in $[0, n_1]$, and so on, until we get a level function in this interval: after this, we apply special shifts in $[n_1 + 1, n]$. As a result, we get $M_0(n_1) = M(n_1)$.

2.4. We have even more:

PROPOSITION 5. *If M satisfies (*), then for each reducing shift Λ, also $\bar{M} = \Lambda M$ satisfies (*).*

Proof. Assume that $\Lambda: n_1 \to n_1 + 1$ destroys property (*). Then we must have

$$\sum_{l=n_1+1}^{n} \bar{m}_l > n - n_1,$$

the sum actually being equal to $n - n_1 + 1$. Thus $\sum_{n_1+1}^{n} m_l = n - n_1$, $m_{n_1} \geq 1$. But since M satisfies (*), we have $m_{n_1} = 1$. It follows from this that M satisfies (*) also on $[0, n_1 - 1]$, and by Proposition 4, $M_{n_1 - 1} = M^0_{n_1 - 1}$. Thus, by an earlier remark, m_k^0 and $\bar{m}_k^0$, $k \geq n_1$ may be computed from the function m_k on $[n_1, n]$. Since this function satisfies the Pólya condition, we have $m_k^0 = 1$, $k \geq n_1$, $\bar{m}_k^0 = 1$, $k > n_1$, $\bar{m}^0_{n_1} = 0$. If M is replaced by $\bar{M}$, the difference $M_k - M_k^0$ will not change for $k \leq n_1$, and can only increase if $k > n_1$. The shift Λ is not reducing. This establishes our result.

A special shift of a function M_1 is also a special shift for $M_1 + M_2$. Hence we obtain:

PROPOSITION 6. *If $M_1 + M_2$ satisfies (*), then also $M_1^0 + M_2$ does.*

2.5. A function M is *decomposable*,

$$M = M_1 \oplus \cdots \oplus M_\mu, \qquad (2.5.1)$$

if there are intervals $[0, k_1], \ldots, [k_{\mu-1}+1, k_\mu]$, $k_\mu = n$ with the property that each of the restrictions M_λ of M to the corresponding interval, has property (*):

$$\sum_{l=k}^{k_\lambda} m(l) \leqslant k_\lambda - k + 1, \qquad k_{\lambda-1} \leqslant k \leqslant k_\lambda, \qquad \lambda = 1, \ldots, \mu. \tag{2.5.2}$$

A reducing shift of M does not cross any of the points k_λ. We have therefore

$$M^0 = M_1{}^0 \oplus \cdots \oplus M_\mu{}^0, \tag{2.5.3}$$

and consequently

$$\alpha(M) = \alpha(M_1) + \cdots + \alpha(M_\mu). \tag{2.5.4}$$

3. Coalescence of Matrices

3.1. In Section 3, we shall disregard the order of rows of matrices. Only when dealing with determinants in Section 4, will this order be essential.

Let E be an $m \times (n+1)$ matrix which is a horizontal submatrix of a Pólya matrix with $(n+1)$ ones. Equivalently, we assume that the Pólya functions $m(k)$, $M(k)$ of E satisfy condition (*). The *coalescence of E to one row form* is the one row matrix E^0 with functions m^0, M^0. Further, let $E = E_1 \cup E_2$ be a decomposition of E into two matrices, formed by two disjoint groups of rows of E. The *coalescence in E of the rows of E_1* is the matrix $E_1{}^0 \cup E_2$; its Pólya function is $M_1{}^0 + M_2$. It follows from Proposition 6 that $E_1{}^0 \cup E_2$ is a Pólya matrix if $E_1 \cup E_2$ is one. From (2.1.5) we obtain

$$(E_1 \cup E_2)^0 = (E_1{}^0 \cup E_2)^0 = (E_1{}^0 \cup E_2{}^0)^0; \tag{3.1.1}$$

and (2.1.6) gives the associative law

$$((E_1 \cup E_2)^0 \cup E_3)^0 = (E_1 \cup (E_2 \cup E_3)^0)^0 = (E_1 \cup E_2 \cup E_3)^0. \tag{3.1.2}$$

For the coefficient of collision we have, from (2.2.2), the relations

$$\alpha(E_1 \cup E_2) = \alpha(E_1) + \alpha(E_2) + \alpha(E_1{}^0 \cup E_2{}^0); \tag{3.1.3}$$

$$\begin{aligned}\alpha(E_1 \cup \cdots \cup E_r) &= \alpha(E_1 \cup E_2) + \alpha((E_1 \cup E_2)^0 \cup E_3) \\ &\quad + \cdots + \alpha((E_1 \cup \cdots \cup E_{r-1})^0 \cup E_r).\end{aligned} \tag{3.1.4}$$

If there is a vertical decomposition of E, $E = E' \oplus E''$, then coalescence of a group of rows in E can be performed by coalescence of the rows of E' and E'', separately.

3.2. In particular, we consider coalescence of two rows E_1, E_2 of E (for example, rows numbered 1 and 2).

Let ρ_k be the function of (2.2.1), and let $\rho_{-1} = 0$. Then $\Delta\rho_k = \rho_k - \rho_{k-1} = m_k - m_k^0$ is equal to 1 exactly when $m_k = 2$, $m_k^0 = 1$; $\Delta\rho_k = -1$ if and only if $m_k = 0$, $m_k^0 = 1$; and $\Delta\rho_k = 0$ for all other values of k. Let k_j, k_j' be positions with $\Delta\rho_k = 1$, $\Delta\rho_k = -1$, respectively. Since $\sum_0^n \Delta\rho_k = 0$, there is an equal number (say p) of k_j, k_j'. Since $\sum_k^n \Delta\rho_l = -\rho_{k-1} \leq 0$, there are, for each k, at least as many $k_j' \geq k$, as there are $k_j \geq k$. Thus, $k_j' > k_j$, $j = 1, \ldots, p$.

The positions k_j are identical with k's for which

$$e_{1k} = e_{2k} = 1; \tag{3.2.1}$$

and the k_j' are among the k's with

$$e_{1k} = e_{2k} = 0; \tag{3.2.2}$$

more precisely, they are exactly the k's with (3.2.2) and $\rho_k \geq 1$.

We thus obtain the Karlin-Karon [3] definition of coalescence:

(A) *k_1' is the first $k_1 > k_1$ with* (3.2.2); *for $j > 1$, k_j' is the first k with $k > k_j$, $k > k'_{j-1}$, which satisfies* (3.2.2).

Let E_1' be the row E_1 with ones in positions k_j replaced by ones at the k_j'. Then $(E_1 \cup E_2)^0$ is the sum of the disjoint rows E_1' and E_2. Row E_1' is *the translation* of E_1 in this coalescence.

Let $l_1 < \cdots < l_q$; $l_1' < \cdots < l_q'$, $l_j \leq l_j'$ be the positions of all ones in rows E_1, E_1', respectively.

By means of Abel's transform we obtain for the coefficient of collision

$$\alpha(E_1 \cup E_2) = \sum_0^n \rho_k = \sum_0^n (n-k)\,\Delta\rho_k$$

$$= \sum_{j=1}^p (k_j' - k_j) = \sum_{j=1}^q (l_j' - l_j). \tag{3.2.3}$$

In analogy to (A) we have for the l_j':

(B) *l_1' is the first $k \geq l_1$ with $m_2(k) = 0$ (or with $e_{2k} = 0$);*
l_j' is the first k with $k > l'_{j-1}$, $k \geq l_j$ and $m_2(k) = 0$.

In variation with Section 2, we define a *shift* A: $k \to k+1$ *of row* E_1 (in coalescence with E_2). This is defined if $e_{1k} = 1$, $e_{1,k+1} = 0$, and moves this one to the position $(k+1)$. Thus, shifts of E_1 produce some, but not all, shifts of the function m. However, if m, that is, $E_1 \cup E_2$, is in collision, then there exists a shift of row E_1 which is a reducing shift of m in the sense of Section 2. For example, let l'_{j_0} be the largest l_j' not an l_i, and l the largest l_i preceding l'_{j_0}, then $l \to l+1$ is the desired shift. Hence the results of Section 2 hold for row shifts.

3.3. *Maximal coalescence.* Let $E = E_1 \cup E_2$ be a matrix satisfying condition (*) of Section 2, let E_1 be a one-row matrix. If $\bar{E}_2$ is a submatrix of E_2 (obtained from E_2 by replacing some of its ones by zeros), then $\bar{M}_2{}^0 \leq M_2{}^0$, and consequently $\bar{E}_2{}^0$ is a subset of $E_2{}^0$. Thus if E_1', $E_1{}^*$ are the translations of row E_1 in the coalescenses of $E_1 \cup \bar{E}_2{}^0$ and $E_1 \cup E_2{}^0$, respectively, the corresponding positions of ones, l_j', $l_j{}^*$, $j = 1,\ldots, q$, will satisfy

$$l_j' \leqslant l_j{}^*, \qquad j = 1,\ldots, q. \tag{3.3.1}$$

Therefore, we call the matrix $E_1{}^* \cup E_2$ the *maximal coalescence* of $E_1 \cup E_2$.

We can find the Pólya function M^* of $E_1{}^*$, if the functions M^0 and M_2 of E^0 and E_2 are known. Since the rows $E_1{}^*$, $E_2{}^0$ are disjoint, $M_1{}^* + M_2{}^0 = M^0$ and

$$M_1{}^*(k) = M^0(k) - M_2{}^0(k), \qquad k = 0,\ldots, n. \tag{3.3.2}$$

In particular, if E is a Pólya matrix,

$$M_1{}^*(k) = k + 1 - M_2{}^0(k), \qquad k = 0,\ldots, n. \tag{3.3.3}$$

For the numbers k_j, l_j ; $k_j{}^*$, $l_j{}^*$ of the coalescence $(E_1{}^* \cup E_2{}^0)^0$,

$$\gamma = \gamma(E_1, E_2) = \sum_{j=1}^{n} (k_j{}^* - k_j) = \sum_{j=1}^{q} (l_j{}^* - l_j) \tag{3.3.4}$$

will be called the coefficient of collision for maximal coalescence.

Its main property is as follows:

PROPOSITION 7. *Let $E_1 \cup E_2$ be a Pólya matrix and let $\tilde{\Lambda}$ be a multiple shift of row E_1 of order $\beta > \gamma$. Then $\tilde{\Lambda}E_1 \cup E_2$ is not a Pólya matrix.*

Proof. If $\tilde{\Lambda}E_1$ has ones in positions $\tilde{l}_j$, then $\beta = \sum (\tilde{l}_j - l_j)$. For some j we must have $\tilde{l}_j > l_j{}^*$; let l be the smallest such $l_j{}^*$. Then the Pólya function $\tilde{M}$ of $\tilde{\Lambda}E_1$ satisfies $M_1{}^*(l) > \tilde{M}(l)$, and for the Pólya function of $\tilde{\Lambda}E_1 + E_2{}^0$ we have

$$\tilde{M}(l) + M_2{}^0(l) < M_1{}^*(l) + M_2{}^0(l) = l + 1.$$

Then also $\tilde{\Lambda}E_1 \cup E_2$ is not a Pólya matrix.

Let E be a matrix with rows $E_1, \ldots, E_m$, and let l_j' be obtained from the translation of row E_1 in the coalescence of E_1 and E_i, $i = 2, \ldots, m$. Then we have (3.3.1). Sometimes more can be said:

PROPOSITION 8. *Let E be a Birkhoff matrix with $e_{10} = 1$. If $l_1' < l_1^*$, then*

$$l_1' < l_2' < l_1^* < l_3' < \cdots < l_q' < l_{q-1}^* < l_q^*. \tag{3.3.5}$$

Proof. We have $l_q^* = n$. Let m' be the Pólya function of the matrix $E' = E_2 \cup \cdots \cup E_m$. We enlarge row E_i to row $\bar{E}_i$ with the function

$$\begin{aligned} \bar{m}_i(k) &= m'^{0}(k) \qquad \text{if} \quad k \neq l_1', \\ &= 0 \qquad\qquad \text{if} \quad k = l_1'. \end{aligned}$$

Then $E_1 \cup \bar{E}_i$ is not a Birkhoff matrix, and the numbers l_j'' of this coalescence satisfy $l_j' \leq l_j''$, $j = 1, \ldots, q$, and $l_q'' < n$. There are exactly $q + 1$ zero points of $\bar{m}_i$, and q zero points of m'^{0}. The latter are occupied by the l_j^*, among the former there are all l_j''. It follows that $l_1'' = l_1'$, $l_2'' = l_1^*, \ldots, l_q'' = l_{q-1}^*$. This gives (3.3.5).

4. DETERMINANTS AND COALESCENCE

Let E be an $m \times (n+1)$ incidence matrix with $(n+1)$ ones. Let $X = (x_1, \ldots, x_m)$ be a set of distinct real or complex knots. The determinant of the Birkhoff interpolation problem (1.1) is given by

$$D_E(X) = |x_i^{n-k}/(n-k)!, \ldots, x_i^{-k}/(-k)!\,;\ e_{ik} = 1|. \tag{4.1}$$

The determinant has $n + 1$ rows given above, corresponding to all pairs (i, k) with $e_{ik} = 1$. We assume that $1/r! = 0$ if $r < 0$. The rows are ordered lexicographically: row (i, k) precedes row (i', k') if and only if $i < i'$, or $i = i'$ and $k < k'$.

For some j, we assume that $z = x_j$ is a free (real or complex) variable, while the x_i, $i \neq j$, are fixed. Let α_i, $i \neq j$, γ be the coefficients of collision in the coalescence of rows j_0, i, and of maximal coalescence. Known results [3, 4, 7] can be summed up as follows:

THEOREM 1. *The determinant $D_E(X)$ is a polynomial in z of degree at most γ, which has zeros of order at least α_i at $z = x_i$, $i \neq j$. The Taylor expansion near x_i is*

$$D_E(X) = \frac{C'}{\alpha_i!} \tilde{D}_{E'}(X')(z - x_i)^{\alpha_i} + \cdots + \frac{C^*}{\gamma!} \tilde{D}_{E^*}(X')(z - x_i)^{\gamma}. \tag{4.2}$$

Here E', E^* are the matrices of coalescence of rows j, i in E, and of maximal coalescence of row j; X' is the set of x_i, $i \neq j$, and ~ over the determinant means that the rows of it inherited from x_j still remain in the old position. The positive integers C', C^* can be identified as numbers of certain (reducing) shifts of row j.

This follows also easily from results of Sections 2 and 3, using the differentiation technique of [7].

In theorems which preserve the order $x_1 < \cdots < x_m$ of the knots, we can coalesce row i only with $i + 1$ or $i - 1$. Considering the first case, we get

$$\tilde{D}_{E'}(X') = (-1)^{\sigma} D_{E'}(X'), \qquad \tilde{D}_{E^*}(X') = (-1)^{\sigma^*} D_{E^*}(X'), \tag{4.3}$$

where σ, σ^* are the *interchange numbers* (defined only modulo 2). For example, let $l_1', ..., l_q'$ be the positions of translated ones of row i under coalescence with row $i + 1$, $\bar{l}_1, ..., \bar{l}_r$ positions of ones of row $i + 1$. Then σ (in coalescence of row i to row $i + 1$) is the number of interchanges needed to transform the sequence

$$l_1', ..., l_q', \bar{l}_1, ..., \bar{l}_r \tag{4.4}$$

into its natural order.

5. Applications to Regularity and Singularity

From Theorem 1 and (4.3) we derive, for coalescence of row i to row $i + 1$ in E,

Lemma 2. *Let $\epsilon = \pm 1$ be the sign of the determinant $D_{E'}(X')$ of the coalesced matrix E' for some set of knots $X' = (x_j, j \neq i)$. Then there exists a set of knots X with $x_i < x_{i+1}$ as close as we wish to x_{i+1}, for which*

$$\operatorname{sign} D_E(x) = (-1)^{\sigma} \epsilon. \tag{5.1}$$

Thus, if E' is strongly singular, E is also strongly singular [3].

A new criterion of singularity is as follows. Let the Pólya matrix E contain a horizontal submatrix consisting of p adjacent rows, $F = F_1 \cup \cdots \cup F_p$. For $p > 2$, F^0 is obtainable in different ways by means of $p - 1$ consecutive coalescences of rows. Consider two such ways, with interchange numbers σ_j, σ_j', and the coefficients α_j, α_j', $j = 1, ..., p - 1$. Because of (3.1.4) we have $\sum_{j=1}^{p-1} \alpha_j = \sum_{j=1}^{p-1} \alpha_j' = \alpha(F) = \alpha$.

Theorem 2. *The matrix E is strongly singular if*

$$\sigma_1 + \cdots + \sigma_{p-1} \neq \sigma_1' + \cdots + \sigma_{p-1}' \pmod 2. \tag{5.2}$$

Proof. By the associativity law (3.1.2), both sequences of coalescences will produce the same matrix $E_{p-1} = E'_{p-1}$, which will satisfy the Pólya condition. We can find a set of knots X_{p-1} with $D_{E_{p-1}}(X_{p-1}) = \epsilon \neq 0$. Applying Lemma 2 several times, we obtain two sets of knots X, X' for which

$$D_E(X) = (-1)^{\Sigma\sigma_j + \lambda}\epsilon, \qquad D_E(X') = (-1)^{\Sigma\sigma_j' + \lambda}\epsilon.$$

One can vary this theorem, by allowing multiple coalescences, maximal coalescences.

As an example, we consider in more detail the case of three rows F_1, F_2, F_3, with ones in positions $l_1', \ldots, l_p'$; $l_1'', \ldots, l_q''$; $l_1''', \ldots, l_r'''$. By $(l_1', \ldots, l_p')_2$ we denote the translation of the row F_1 in coalescence $(F_1 \cup F_2)^0$, and similarly for other coalescences.

We show that we may first perform all translations, then all interchanges. In the coalescence $((F_1 \cup F_2)^0 \cup F_3)^0$ we have as the final set of ones

$$((l_1', \ldots, l_p')_2, l_1'', \ldots, l_q'')_3, l_1''', \ldots, l_r'''. \tag{5.3}$$

while coalescence $(F_1 \cup (F_2 \cup F_3)^0)^0$ leads via $F_{23} = (F_2 \cup F_3)^0$ to the sequence

$$(l_1', \ldots, l_p')_{23}, (l_1'', \ldots, l_q'')_3, l_1''', \ldots, l_r'''. \tag{5.4}$$

We have to compare $\sigma = \sigma_1 + \sigma_2$ with $\sigma' = \sigma_1' + \sigma_2'$, where σ_1 is the number of interchanges for $(l_1', \ldots, l_p')_2, l_1'', \ldots, l_q''$ which brings it to natural order, or equivalently, the number of interchanges for the sequence

$$((l_1', \ldots, l_p')_2, l_1'', \ldots, l_q'')_3, \tag{A}$$

and σ_2 is the number of interchanges for the sequence consisting of *ordered* sequence (A), followed by $l_1''', \ldots, l_r'''$. Thus, σ is congruent mod 2 to the number of interchanges of (5.3) which bring it in increasing order. Similarly, σ' is the number of interchanges for (5.4). The two sequences consist of the same integers because of the associativity law of coalescence. The difference $\Delta = \sigma - \sigma'$ is congruent to the number of interchanges which transform (5.3) into (5.4). Omitting the l_j''' at the end, we have

THEOREM 3. *The matrix E is singular if it contains three rows for which sequences* (A) *and*

$$(l_1', \ldots, l_p')_{23}, (l_1'', \ldots, l_q'')_3 \tag{B}$$

belong to different permutation classes.

EXAMPLE 1. If a matrix has three adjacent rows with ones in positions

5, 6; 5, 7 and 6, 7, or if it has three rows with ones in positions 4; 5 and 4, it is singular. In the first case, the sequences (A) and (B) are, respectively,

$$((5, 6)_2, 5, 7)_3 = (6, 8, 5, 7)_3 = 8, 10, 5, 9,$$
$$(5, 6)_{23}, (5, 7)_3 = 9, 10, (5, 7)_3 = 9, 10, 5, 8.$$

EXAMPLE 2. Assume that row F_1 consists of a single 1 in position k; the portion of F_2 with $k \leqslant p + 1$ consists of one in position p, while the portion F_2' of F_2 with $k > p + 1$ is arbitrary; F_3 consists of ones in positions $0 \leqslant k < p$. Let $k_0 < p$. The sequences of Theorem 3 are

$$p, \quad p + 1; \quad F_2'; \tag{A}$$
$$p + 1, \quad p; \quad F_2'. \tag{B}$$

Thus, each matrix E, containing three adjacent rows F_1, F_2, F_3, is strongly singular.

EXAMPLE 3. Let the matrix E consist of three rows, with ones in positions $i = 1$, $0 \leqslant k \leqslant p$; $i = 2$, $k = k_1, k_2$; $i = 3$, $0 \leqslant k \leqslant q$, where $k_1 \leqslant p \leqslant q \leqslant k_2$, $n = p + q + 1$. The sequences of Theorem 3 are here:

$$*q, q + 1, \ldots, q + p*, \; q + k_1, \; q + p + 1 \tag{A}$$

(where between $* \cdots *$ the term $q + k_1$ is omitted), $\sigma = p - k_1$.

$$\begin{aligned} &(a) \quad q + 1, \ldots, k_2 - 1, k_2 + 1, \ldots, q + p + 1, q, k_2 \qquad \text{if } k_2 > q, \\ &(b) \quad q + 2, \ldots, p + q + 1, q, q + 1 \qquad \text{if } k_2 = q, \end{aligned} \tag{B}$$

so that $\sigma' = p + n - k_2$ in case (a), $\sigma' = 2p$ in case (b). Hence $\Delta = n - k_1 - k_2$ in case (a), $\Delta = n - k_1 - k_2 - 1$ in case (b).

Result: if $n - k_1 - k_2$ is odd, and $k_2 > q$, or if $n - k_1 - k_2$ is even and $k_2 = q$, then the matrix E is *order*-singular.

As another corollary of Theorem 2 we have: Let E have a row F^0, where $F^0 = F_1 \cup \cdots \cup F_p$, is as in Theorem 2. Then E is singular.

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